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A simple novel constitutive equation for concrete

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A simple novel constitutive equation, three parameter strength criterion for concrete is proposed to represent the composite nature and complex failure mechanism of material of concrete. In this paper, the study is to demonstrate the use of scalar valued function, invariant theory, as applied to concrete failure prediction. Without the loss of accuracy of prediction, a three parameter strength criterion is developed.

1. Introduction

The characteristic properties of concrete have been shown to be those of a complex, multiphase material which is best studied as a composite. The physical properties in the final state (hydrated state) depend on the original mixed proportions and the environmental conditions during cure. Real concrete is, in general, non-homogeneous, anisotropic and non-continuous, as it is composed of groups of elements formed into a large number of discrete particles. However, there is a dimensional level of aggregation (the phenomenological or engineering level) at which the concept of the structural element can be replaced by a homogeneous, isotropic, continuous medium composed of structural elements of identical properties. The mechanical characteristics of concrete are best idealized at the macroscopic level for engineering design applications. The assumption of homogeneity can be justified only on a statistical basis after consideration of the average properties of the elements in the body.

The failure for concrete has been shown to be initiated by numerous microscopic flaws or cracks inherent within the concrete matrix. The average influence of these microscopic flaws, as viewed from macroscopic theory, reveal distinct levels of change in the mechanical behaviour of concrete. As the stress level increases, the mechanical behaviour changes from quasi-elastic to plastic, with two distinct points of departure. The initial discontinuity begins at the onset of stable fracture propagation, while the ultimate strength is reached at the onset of unstable fracture propagation. The hydrostatic (spherical) and deviatoric components of the localized stress have been shown to delay and propagate the internal crack growth, respectively.

The development of a strength criterion for a material depends on its stress state at or during failure conditions, either it is brittle or ductile. Consideration of mechanical response and failure mode shows that concrete is best classified as a brittle material for normal hydrostatic pressure. The strength characterization of most brittle materials depends on the hydrostatic as well as the deviatoric component of stress, while the characterization of the ductile material is independent of the hydrostatic component.

Most strength criteria presented in previous papers follow functional forms, which are functions of stress tensors. The strength criteria presented in these papers show poor agreement with experimental results. The theories presented often depend on the material co-ordinate systems, and are not invariant, and so require complex methodologies for characterization of material parameters. These criteria have, for the most part, been formulated within the framework of classical theories of plasticity, which are subject to a number of strong constraints. These approaches lack generality and agreement with physical laws.

In recent years, as complex, anisotropic, fibre reinforced composites have been manufactured and used as structural elements, more appropriate methods for the characteristics of materials have been sought. In the field of non-linear continuum mechanics there have been continuous developments following more powerful approaches to these problems. In reviewing the recently proposed general strength criteria, the continuum mechanics approach has been most prominent. The application of general and explicit tensor based scalar-valued or tensor-valued functions has proven to be useful for developing strength criteria and constitutive equations. Many investigations have shown the value of using tensor function theory in these applications.

The composite nature and complex failure mechanism of concrete dictate a need for a more rational approach to strength criterion development. Use of the invariant strength function theory, a six parameter constitutive equation for concrete have been obtained by the author. For the purpose of simplicity in this paper, a simple novel constitutive equation, three parameter, is proposed without loss of significant engineering accuracy. Although the six parameter constitutive equation yields the highest accuracy, the simple three parameter constitutive equation also can predict accurately the strength of concrete.

2. Experimental procedure

2.1. Development of the simple novel strength criterion

The development of a strength criterion for the prediction of the ultimate strength of concrete under multiaxial loadings should be formulated from the systematic theories of modern continuum mechanics. The criterion should be validated by accurate experimental data for the determination of the failure surface for concrete. A strength criterion to predict the failure of concrete is by necessity governed by the failure mechanisms. These failure mechanisms must be related mathematically, and yield a failure surface in a stress space.

The tensor functional technique of non-linear continuum mechanics is the most logically applicable to the formulation of the strength criterion. Such functionals satisfy the requirement of invariance for a group of orthogonal transformations specific to the material symmetry. In addition, tensor function theory allows inclusion of any number of stress interaction terms, which gives the theory a broad applicability to the characterization of anisotropic material. The tensor functional technique for the development of a strength criterion is a new approach which produces a rational criterion.

A strength function has been shown to be expressible as

$$f(\sigma_{ij}) = 1 \qquad i, j = 1, 2, 3 \tag{1}$$

where σ_{ij} is a stress tensor referred to an arbitrary co-ordinate system. The form of the failure function in Equation 1 has been followed by past investigators. In general, the strength criteria presented were functions of the applied stress which were non-invariant, i.e. [1–4] etc.

A strength function for a given material symmetry (isotropic for concrete) must be invariant under a complete point group of transformations of co-ordinates, $\{t_{ij}\}$, which associate with the group of material symmetry. This insures that the strength criterion is a scalar (invariant under the appropriate group of co-ordinate transformation), and is a single-valued function, as indicated by Equation 1. It is known that failure is a physical phenomenon which is totally independent of co-ordinates. Thus the requirement of invariance states

$$f(\bar{\sigma}_{ij}) = f(\sigma_{ij}) \qquad i, j = 1, 2, 3 \tag{2}$$

where $(\bar{\sigma}_{ij})$ represents the transformed stress components

$$\bar{\sigma}_{ij} = t_{ir} t_{js} \sigma_{rs} \qquad i, j, r, s = 1, 2, 3 \tag{3}$$

Invariant quantities for each class of anisotropic materials have been obtained [5, 6]. Huang [6] determined the second-, fourth- and sixth-order of invariant quantities in the three-dimensional case for each of the crystal classes from consideration of invariant transformations of the strength function. The invariant quantities for the isotropic material symmetry case are as follows

$$\begin{aligned} I_1^1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2^2 &= -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ I_3^3 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{4}$$

where I_j^i is the jth invariant quantities of ith degree.

The strength function for an isotropic material in the form of Equation 1 can be rewritten as

$$f(I_1, I_2, I_3) = 1 \tag{5}$$

Also, Equation 5 can be expressed in terms of the deviatoric and spherical invariant quantities, where

$$\begin{aligned} J_2 &= 1/6[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \\ J_3 &= (\sigma_1 - \sigma)(\sigma_2 - \sigma)(\sigma_3 - \sigma) \\ \sigma &= \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_1 \end{aligned} \tag{6}$$

Therefore, a strength function is also expressible as a function

$$f(I_1, J_2, J_3) = 1 \tag{7}$$

The proposed strength function by Chen and Chen [7] followed the invariant of a tensor function of a second degree. A two equation strength criterion, using the invariant quantities of Equation 7, is given as

$$f(I_1, J_2) = J_2 + \frac{A_u}{3} I_1 = t_u^2 \tag{8a}$$

for the compression–compression region, and for all other regions, (compression–tension, tension–tension and tension–compression) as

$$f(I_1, J_2) = J_2 - \frac{1}{6} I_1^2 + \frac{A_u}{3} I_1 = t_u^2 \tag{8b}$$

where A_u and t_u^2 are material parameters.

The strength functions of Equation 8 are a special form of linear combination of invariants. The functions are of quadratic form. The quadratic form has been addressed and shown to be inadequate in its definition of the failure envelope for the biaxial principal stress plane. The cubical forms of the polynomial based on tensor function theory have been discussed [6, 8–10]. They suggested that the third degree terms are necessary to be included and to explain the additional stress interaction relations. The quadratic form at best can describe a conic curve which may not yield accurate correlations with experimental data for concrete in all four quadrants of the biaxial plane.

In order to formulate a cubic strength function, the invariant quantities of I_j^3 must be included. Thus the invariant quantities of each degree for an isotropic material are

$$I^1\text{, first degree: } I_1$$

$$I^2\text{, second degree: } I_1^2,\ J_2$$

$$I^3\text{, third degree: } I_1^3,\ I_1J_2,\ J_3,\ I_1^2(J_2)^{1/2} \tag{9}$$

The system of quantities, Equation 9, represents terms which are required to form a cubic strength function for isotropic materials.

A strength function, which is a combination of the invariant quantities, Equation 9, has been proposed for concrete by Huang [11], as follows

$$\mathrm{A}_1 I_1 + \mathrm{A}_{11} I_1^2 + \mathrm{A}_{22} J_2 + \mathrm{A}_{111} I_1^3 + \mathrm{A}_{122} I_1 J_2 \mp \mathrm{A}_{112} I_1^2 (J_2)^{1/2} = 1 \tag{10}$$

where all As are material parameters of the strength tensor.

The six independent material parameters (A_1, A_{11}, A_{111}, A_{22}, A_{122}, A_{112}) are determined from experimental strength data for concrete, and thus characterize the strength quality of concrete. However, at least six independent engineering tests are required for the determination of Equation 10. For engineering purposes, a novel strength criterion should be established by using the least number of engineering tests which is enough to produce a criterion of acceptable accuracy. Therefore, this study has sought to simplify the proposed Equation 10 through the elimination of terms with slight influence on the material strength. From comparison of the strength criteria, a simple, three parameter strength function is judiciously proposed. This proposed function is similar to that of Chen and Chen [7], but it includes an additional term of higher order (cubic tensor) invariant. The material parameters A_1, A_{11}, A_{22} and A_{111} can be expressed in terms of three new material constants a_1, A and B^2. The proposed strength function for concrete subjected to biaxial states of stresses is given as follows

$$f(I_j^i) = \frac{\mathrm{a}_1}{3} I_1 + \mathrm{A} I_1^2 + \frac{\mathrm{a}_3}{27} I_1^3 + J_2 = \mathrm{B}^2 \tag{11}$$

where a_1, a_3 and B^2 are material parameters which are determined through simple engineering material tests, and A is a constant value. Equation 11 represents a simple strength function which satisfies the invariant requirement of isotropic material symmetry. This form is analytically simpler and more physically vigorous than previously proposed criteria which were based on classical plasticity approaches to fracture definition. In addition, the proposed criterion represents a novel, complete and unified function capable of accurately describing the biaxial failure envelope.

3. Discussion

3.1. Characterization coefficients of the novel strength criterion

The proposed strength function presented in Equation 11 is characterized to a given strength quality of concrete by the three material parameters, a_1, a_2 and B^2. The three parameters are determined from simple engineering material tests. The engineering material tests required are: uniaxial compressive strength (f_c), uniaxial tensile strength (f_t), and the biaxial spherical compressive strength (f_{bc}).

To determine the three material parameters, the proposed strength function, Equation 11, is rewritten in terms of the stress states within the test specimen during the tests. By solving the following three simultaneous independent equations, the parameters are determined.

1. Uniaxial compression

$$\sigma_1 = -f_c;\ \sigma_2 = \sigma_3 = 0$$

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 = -f_c$$

$$J_2 = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{f_c^2}{3}$$

substituting I_1 and J_2 into Equation 11 yields

$$\frac{-\mathrm{a}_1}{3} f_c + \mathrm{A} f_c^2 - \frac{\mathrm{a}_3}{27} f_c^3 + \frac{f_c^2}{3} = \mathrm{B}^2 \tag{12}$$

2. Uniaxial tension

$$\sigma_1 = f_t;\ \sigma_2 = \sigma_3 = 0$$

$$I_1 = f_t$$

$$J_2 = \frac{f_t}{3}$$

substituting I_1 and J_2 into Equation 11 yields

$$\frac{\mathrm{a}_1}{3} f_t + \mathrm{A} f_t^2 + \frac{\mathrm{a}_3}{27} f_t^3 + \frac{f_t^2}{3} = \mathrm{B}^2 \tag{13}$$

3. Biaxial spherical compression

$$\sigma_1 = \sigma_2 = -f_{bc};\ \sigma_3 = 0$$

$$I_1 = -2f_{bc}$$

$$J_2 = \frac{f_{bc}}{3}$$

substituting I_1 and J_2 into Equation 11 yields

$$\frac{-2\mathrm{a}_1}{3} f_{bc} + 4\mathrm{A} f_{bc}^2 - \frac{8\mathrm{a}_3}{27} f_{bc}^3 + \frac{f_{bc}^3}{3} = \mathrm{B}^2 \tag{14}$$

Equations 12, 13 and 14 yield three equations with three unknown material parameters. The three simultaneous linear equations are solved and the three parameters are given in terms of the simple material strengths as

$$\mathrm{a}_1 = \frac{[(3\mathrm{A}+1)(f_t^2 - f_c^2)(8f_{bc}^3 - f_c^3) - 3\mathrm{A}(f_c^2 - 4f_{bc}^2)(f_t^2 + f_c^3) - (f_c^2 - f_{bc}^2)(f_t^2 + f_c^3)]}{(2f_{bc} - f_c)(f_t^3 + f_c^3) - (f_t + f_c)(8f_{bc}^3 - f_c^3)} \tag{15}$$

$$a_3 = \frac{-a_1(f_t + f_c) - 3(A + 1/3)(f_t^2 - f_c^2)}{(f_t^3 + f_c^3)} \tag{16}$$

$$B^2 = Af_c^2 - \frac{a_3 f_c^3}{27} + \frac{f_c^2}{3} - \frac{a_1 f_c}{3} \tag{17}$$

where f_c, f_t and f_{bc} are absolute material property values. Therefore by measuring or estimating the three material strengths, f_c, f_t and f_{bc}, the strength of concrete is completely characterized.

The value of the constant A in Equation 11 is a constant and independent of property of materials, but does have an important influence upon the shape of the failure envelope. Determination of the value A has been accomplished by comparison of the failure envelope generated from Equation 11 to the failure envelope implied through experimental data. The regions of stress state influenced to the greatest extent by the value have been found to be the tension–tension (T–T) and tension–compression (C–T) quadrants of the principal stress plane. The value of the constant A which yields the highest accuracy within these regions has been determined to be

$$A = +0.2 \text{ in compression–compression region}$$

and

$$A = -0.34 \text{ in tension–tension and tension–compression regions} \tag{18}$$

The increasing accuracy of the strength criterion, by the change of the value of constant A in the C–C region from -0.34 to $+0.2$, is demonstrated by comparison with experimental data. The accuracy of the strength function to conform with the failure envelopes is shown very well as in the following Figs 1, 2 and 3. The advantage of the proposed strength criterion for concrete is mathematically simple and physically sound. It proves to be more accurate than the other criteria, with exception to the complete cubic strength function, Equation 10.

The experimental failure data for concrete of various strengths of Kupfer *et al.* [13] are used for evaluation of the strength envelopes of Equations 10 and 11 on the concrete. The investigators [13] tested three different compressive strengths of concrete, 1.8×10^6, 3.1×10^6 and $5.8 \times 10^6\,\text{kg m}^{-2}$. The material parameters for the proposed simple novel strength criterion (three parameter Equation 13 are given in Table I, and the results for the cubic strength criterion [11] (six parameter Equation 10), are also given in Table II.

For comparison, the results yield by the complete cubic strength function, Equation 10, the criterion proposed by Chen and Chen [7], and the proposed simplified three parameter strength criterion, Equation 11, are plotted in Figs 1, 2 and 3 for various strengths of concrete. The complete cubic function given in [11] proves to be of higher accuracy than others. The biaxial failure envelope generated by the complete cubic function complies with the experimentally determined envelope exceptionally well. In

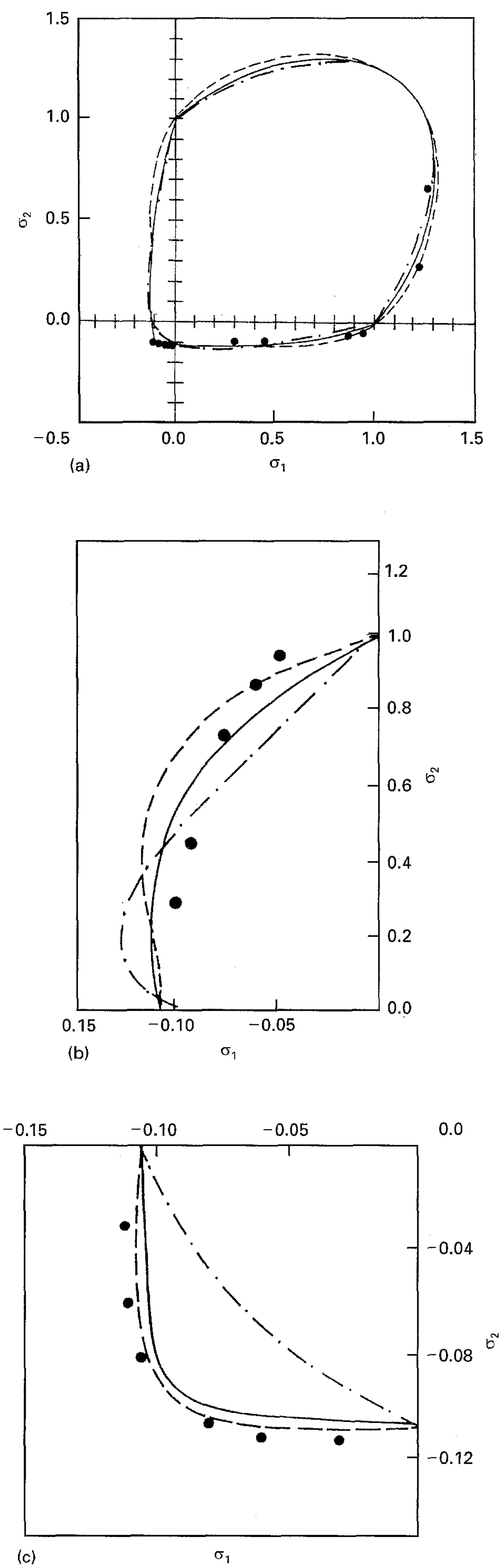

Figure 1 Strength surface for (a) $\sigma_c = 1.8 \times 10^6\,\text{kg m}^{-2}$, (b) C–T region, and (c) T–T region: (——) modified cubic function, (– – –) cubic function, (—·—) after [7].

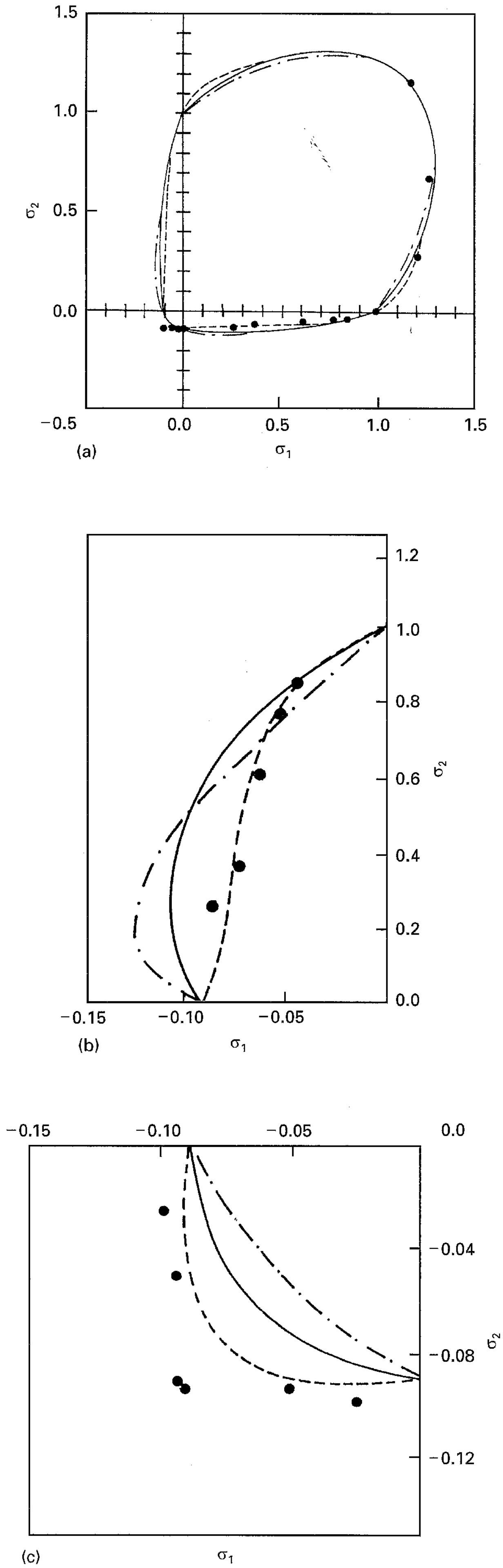

Figure 2 Strength surface for (a) $\sigma_c = 3.1 \times 10^6\ \mathrm{kg\,m^{-2}}$, (b) C–T region, and (c) T–T region: (——) modified cubic function, (– – –) cubic function, (—·—) after [7].

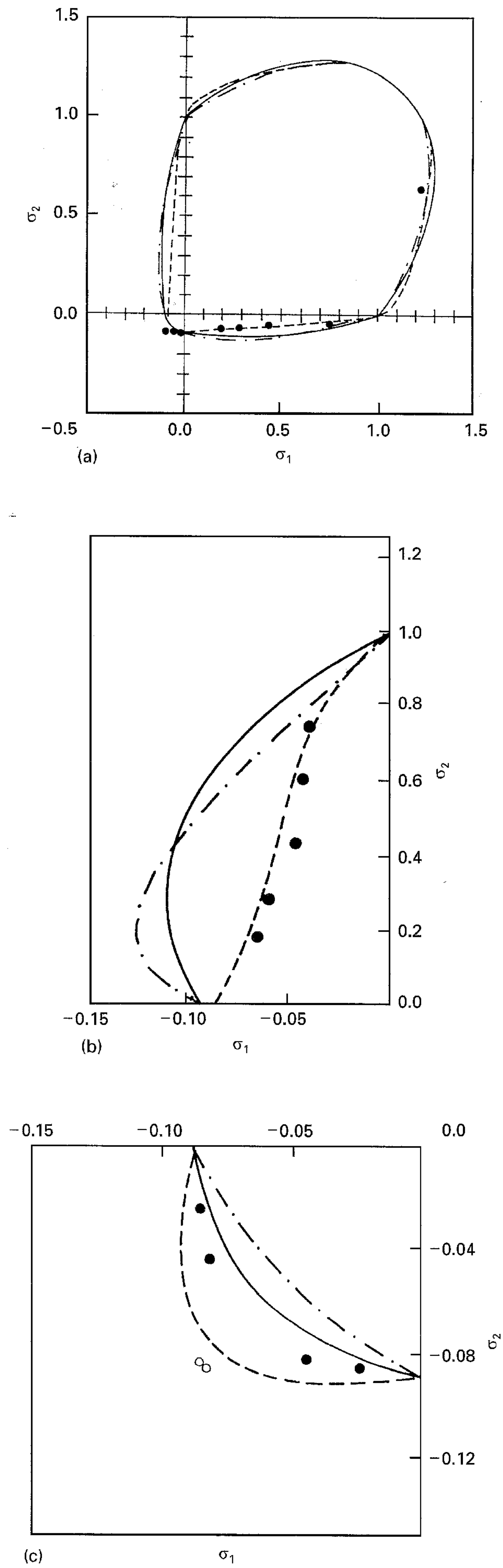

Figure 3 Strength surface for (a) $\sigma_c = 5.8 \times 10^6\ \mathrm{kg\,m^{-2}}$, (b) C–T region, and (c) T–T region: (——) modified cubic function, (– – –) cubic function, (—·—) after [7].

TABLE I Three parameter criterion coefficients

Concrete strength ($kg\,m^{-2}$)	A_1[a]	A_{11}[b]	A_{22}[c]	A_{111}[d]
For T–T and T–C (A = − 0.34)				
1.8×10^6	11.274	− 33.028	97.142	− 12.922
3.1×10^6	11.274	− 33.028	97.142	− 12.922
5.8×10^6	11.272	− 31.735	93.338	− 12.894
For C–C (A = 0.2)				
1.8×10^6	10.043	4.425	22.123	0.756
3.1×10^6	10.043	4.425	22.123	0.756
5.8×10^6	10.047	4.409	22.045	0.710

[a] $A_1 = a_1/3B^2$.
[b] $A_{11} = A/B^2$.
[c] $A_{22} = a_3/27B^2$.
[d] $A_{111} = 1/B^2$.

TABLE II Six parameter criterion coefficients

Concrete strength ($kg\,m^{-2}$)	A_1	A_{11}	A_{22}	A_{111}	A_{122}	A_{112}
1.8×10^6	− 8.762	− 33.923	89.000	− 6.165	− 110.481	98.745
3.1×10^6	− 8.729	− 39.995	138.408	− 17.500	− 250.285	181.025
5.8×10^6	− 8.351	− 43.690	164.366	− 22.948	− 328.088	226.142

order to demonstrate its capabilities, the regions of tension–tension and compression–tension are also enlarged in the figures. The function fit to the test data points is the most superior. However, the proposed three parameter function of this study given in Equation 11 proves also very accurate. It shows to be only slightly less accurate than the complete six parameter function. Indeed within the region of greatest interest, compression–compression, the three parameter function is accurate as equally as the complete cubic strength criterion. It is really simple, also it is physically and mathematically sound. Only the basic engineering material tests are required, such as uniaxial compressive strength, uniaxial tensile strength and the biaxial compressive strength. It is truly convenient for practical engineers to use.

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